Тренировочные задачи по УМФ для подготовки к экзамену,

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I) Решить методом Фурье задачи для уравнения теплопроводности:

1) $$\begin{cases} u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = x(1-x), \end{cases} \qquad u = u(x,t) = ?$$

2) $$\begin{cases} u_t = 4u_{xx}, \quad 0 < x < 2\pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u(2\pi,t) = 0, \\ u(x,0) = 1, \end{cases} \qquad u = u(x,t) = ?$$

3) $$\begin{cases} u_t = u_{xx} - u, \quad 0 < x < 6, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(6,t) = 0, \\ u(x,0) = 3 - x, \end{cases} \qquad u = u(x,t) = ?$$

4) $$\begin{cases} u_t = 2u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = \sin^2 x + \cos^2 3x, \end{cases} \qquad u = u(x,t) = ?$$

5) $$\begin{cases} u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = 16\cos^4 x, \end{cases} \qquad u = u(x,t) = ?$$

6) $$\begin{cases} u_t = u_{xx} - 2u - \sin\dfrac{x}{2}, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = \sin\dfrac{3x}{2}, \end{cases} \qquad u = u(x,t) = ?$$

7) $$\begin{cases} u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0, \\ u(0,t) = 0, \quad u(2,t) = 0, \\ u(x,0) = \sin^3 x, \end{cases} \qquad u = u(x,t) = ?$$

8) $\begin{cases} u_t = 4u_{xx}, \quad 0 < x < 3\pi, \quad t > 0, \\ u(0,t) = 0, \quad u(3\pi,t) = 0, \\ u(x,0) = 3\pi, \end{cases}$ $\qquad u = u(x,t) = ?$

9) $\begin{cases} u_t = u_{xx}, \quad 0 < x < 4, \quad t > 0, \\ u(0,t) = 0, \quad u_x(4,t) = 0, \\ u(x,0) = 2\sin\dfrac{4\pi x}{8}\cos\dfrac{2\pi x}{8}, \end{cases}$ $\qquad u = u(x,t) = ?$

10) $\begin{cases} u_t = u_{xx} + \sin\pi x, \quad 0 < x < 3, \quad t > 0, \\ u(0,t) = 0, \quad u(3,t) = 0, \\ u(x,0) = 0, \end{cases}$ $\qquad u = u(x,t) = ?$

II) Решить методом Фурье задачи для уравнения колебаний струны:

1) $\begin{cases} u_{tt} = 4u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = 1, \end{cases}$ $\qquad u = u(x,t) = ?$

2) $\begin{cases} u_{tt} = u_{xx} - 2u, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \\ u(x,0) = \sin 2x, \quad u_t(x,0) = \sin x, \end{cases}$ $\qquad u = u(x,t) = ?$

3) $\begin{cases} u_{tt} = 4u_{xx} - 4\sin 5x, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \end{cases}$ $\qquad u = u(x,t) = ?$

4) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < 10, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(10,t) = 0, \\ u(x,0) = 10 - x, \quad u_t(x,0) = 1, \end{cases}$ $\qquad u = u(x,t) = ?$

5) $\begin{cases} u_{tt} = u_{xx} - u, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = \cos^2 x, \quad u_t(x,0) = \sin^2 x, \end{cases}$ $\qquad u = u(x,t) = ?$

6) $\begin{cases} u_{tt} = u_{xx} - 4t\sin^2 3x, \quad 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \end{cases}$ $u = u(x,t) = ?$

7) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = \sin\dfrac{3x}{2}, \quad u_t(x,0) = 2\sin\dfrac{x}{2}, \end{cases}$ $u = u(x,t) = ?$

8) $\begin{cases} u_{tt} = 4u_{xx}, \quad 0 < x < 2, \quad t > 0, \\ u(0,t) = 0, \quad u_x(2,t) = 0, \\ u(x,0) = x, \quad u_t(x,0) = -x, \end{cases}$ $u = u(x,t) = ?$

9) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = 3\sin 2\pi x, \quad u_t(x,0) = x, \end{cases}$ $u = u(x,t) = ?$

10) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < 2\pi, \quad t > 0, \\ u_x(0,t) = 0, \quad u(2\pi,t) = 0, \\ u(x,0) = \cos^3\dfrac{x}{4}, \quad u_t(x,0) = 0, \end{cases}$ $u = u(x,t) = ?$

III) Найти непрерывные решения для следующих задач на полупрямой:

1) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(0,t) = t + 1, \\ u(x,0) = 1, \quad u_t(x,0) = x + 1, \end{cases}$ $u = u(x,t) = ?$

2) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(0,t) = \sin 5t, \\ u(x,0) = 0, \quad u_t(x,0) = 5\cos 5x, \end{cases}$ $u = u(x,t) = ?$

3) $\begin{cases} u_{tt} = 25u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(0,t) = \sin 5t, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \end{cases}$ $u = u(x,t) = ?$

4) $\begin{cases} u_{tt} = 4u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(0,t) = 0, \\ u(x,0) = x^3, \quad u_t(x,0) = 0, \end{cases}$ $\qquad u = u(x,t) = ?$

5) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u_x(0,t) = 1, \\ u(x,0) = 2x, \quad u_t(x,0) = 0, \end{cases}$ $\qquad u = u(x,t) = ?$

6) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u_x(0,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = \sin x, \end{cases}$ $\qquad u = u(x,t) = ?$

7) $\begin{cases} u_{tt} = 9u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u_x(0,t) = t, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \end{cases}$ $\qquad u = u(x,t) = ?$

8) $\begin{cases} u_{tt} = 100u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u_x(0,t) = t + 1, \\ u(x,0) = x + 1, \quad u_t(x,0) = x + 1, \end{cases}$ $\qquad u = u(x,t) = ?$

9) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(0,t) = 1, \\ u(x,0) = 1, \quad u_t(x,0) = 1, \end{cases}$ $\qquad u = u(x,t) = ?$

10) $\begin{cases} u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u_x(0,t) = 0, \\ u(x,0) = x, \quad u_t(x,0) = 0, \end{cases}$ $\qquad u = u(x,t) = ?$

IV) В круге $x^2 + y^2 < R^2$ найти гармоническую функцию, если:

1) $R = 5, \qquad u\big|_{x^2+y^2=25} = x^2,$ $\qquad u = u(x,y) = ?$

2) $R = 4, \qquad u\big|_{x^2+y^2=16} = (x+y)^2,$ $\qquad u = u(x,y) = ?$

3) $R = 3, \qquad u\big|_{x^2+y^2=9} = x^2 - 2xy - 4y^2,$ $\qquad u = u(x,y) = ?$

4) $R = 2, \qquad u\big|_{x^2+y^2=4} = 1 + x + x^2 + x^3,$ $\qquad u = u(x,y) = ?$

5) $R=1,\quad u\big|_{x^2+y^2=1}=y^4,\qquad u=u(x,y)=?$

V) При $x^2+y^2>R^2$ найти ограниченную гармоническую функцию, если:

1) $R=5,\quad u\big|_{x^2+y^2=25}=x,\qquad u=u(x,y)=?$

2) $R=4,\quad u\big|_{x^2+y^2=16}=1-4y,\qquad u=u(x,y)=?$

3) $R=3,\quad u\big|_{x^2+y^2=9}=x^2,\qquad u=u(x,y)=?$

4) $R=2,\quad u\big|_{x^2+y^2=4}=(x+y)^2,\qquad u=u(x,y)=?$

5) $R=1,\quad u\big|_{x^2+y^2=1}=x^3,\qquad u=u(x,y)=?$

VI) В кольце $R_1^2<x^2+y^2<R_2^2$ найти гармоническую функцию, если:

1) $R_1=1,\quad R_2=2,\quad u\big|_{x^2+y^2=1}=1,\quad u\big|_{x^2+y^2=4}=2,\qquad u=u(x,y)=?$

2) $R_1=1,\quad R_2=2,\quad u\big|_{x^2+y^2=1}=2,\quad u\big|_{x^2+y^2=4}=1,\qquad u=u(x,y)=?$

3) $R_1=1,\quad R_2=3,\quad u\big|_{x^2+y^2=1}=1,\quad u\big|_{x^2+y^2=9}=x+y,\qquad u=u(x,y)=?$

4) $R_1=2,\quad R_2=3,\quad u\big|_{x^2+y^2=4}=x,\quad u\big|_{x^2+y^2=9}=y,\qquad u=u(x,y)=?$

5) $R_1=1,\quad R_2=5,\quad u\big|_{x^2+y^2=1}=0,\quad u\big|_{x^2+y^2=25}=x^2,\qquad u=u(x,y)=?$

Дополнительные задачи

VII) При помощи интеграла Пуассона решить задачу Коши для уравнения теплопроводности

$$\begin{cases} u_t = a^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), \end{cases} \qquad u = u(x,t) = ?$$

Выразить ответ через функцию ошибок $\operatorname{erf} z = \dfrac{2}{\sqrt{\pi}} \displaystyle\int\limits_0^z e^{-s^2}\, ds.$

1) $a^2 = 1, \qquad u_0(x) = \begin{cases} 0, & x < 0, \\ 2, & x > 0. \end{cases}$

2) $a^2 = 4, \qquad u_0(x) = \begin{cases} 4, & x < 2, \\ 0, & x > 2. \end{cases}$

3) $a^2 = 25, \qquad u_0(x) = \begin{cases} 3, & x < 7, \\ -1, & x > 7. \end{cases}$

4) $a^2 = 1, \qquad u_0(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$

5) $a^2 = 9, \qquad u_0(x) = \begin{cases} 0, & |x| < 4, \\ 6, & |x| > 4. \end{cases}$

6) $a^2 = 1/4, \qquad u_0(x) = \begin{cases} -2, & x < 1, \\ 0, & 1 < x < 10, \\ 8, & x > 10. \end{cases}$

VIII) При помощи формулы Даламбера решить задачу Коши для уравнения колебаний струны

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad x \in \mathbb{R}, \quad t \geqslant 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \end{cases} \qquad u = u(x,t) = ?$$

Записать ответ со всеми возможными упрощениями.

1) $a^2 = 1, \qquad \varphi(x) = e^x, \qquad \psi(x) = e^{2x}.$

2) $a^2 = 4, \qquad \varphi(x) = x^2, \qquad \psi(x) = 1 + x.$

3) $a^2 = 5, \qquad \varphi(x) = 1 - x^2, \qquad \psi(x) = 1 + x^2.$

4) $a^2 = 9, \qquad \varphi(x) = \sin 2x, \qquad \psi(x) = -1.$

5) $a^2 = 1, \qquad \varphi(x) = 0, \qquad \psi(x) = \cos x + 6\cos 3x.$

6) $a^2 = 4, \qquad \varphi(x) = x, \qquad \psi(x) = xe^{-x}.$

IX) Найти сферически симметричное решение $u = u(r)$ для следующей задачи в $\mathbb{R}^n$.

1) $n = 3, \quad 1 < r < 2, \quad \Delta u(r) = 0, \quad u(1) = 0, \quad u(2) = 1.$

2) $n = 3, \quad 1 < r < 2, \quad \Delta u(r) = 0, \quad u(1) = 1, \quad u(2) = 0.$

3) $n = 4, \quad 2 < r < 4, \quad \Delta u(r) = 0, \quad u(2) = 10, \quad u(2) = -10.$

4) $n = 4, \quad 1 < r < 3, \quad \Delta u(r) = 1, \quad u(1) = 0, \quad u(2) = 0.$

5) $n = 5, \quad 0 \leqslant r < 1, \quad \Delta u(r) = 1, \quad u(1) = 10.$

6) $n = 6, \quad 0 \leqslant r < 2, \quad \Delta u(r) = r^2, \quad u(2) = -1.$